



# Observer-less output-feedback global continuous control for the finite-time and exponential stabilization of mechanical systems with constrained inputs



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## ARTICLE INFO

### Article history:

Received 24 September 2016

Revised 21 February 2017

Accepted 20 April 2017

Available online 27 April 2017

Recommended by F. Mazenc

### Keywords:

Finite-time stabilization

Local homogeneity

Output feedback

Mechanical systems

Constrained inputs

Saturation

## ABSTRACT

An observer-less output-feedback global continuous control scheme for the finite-time or (local) exponential stabilization of mechanical systems with constrained inputs is proposed. The approach is formally developed within the theoretical framework of local homogeneity. The closed-loop analysis incorporates a complementary insight on the control-induced motion dissipation through an *ad hoc* feedback-system passivity theorem. The work includes a simulation implementation section where the performance difference of the proposed scheme with previous observer-based and differentiation algorithms is brought to the fore.

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## 1. Introduction

The last decades have witnessed an increasing interest on stabilization with finite-time convergence through continuous feedback. Such an intriguing topic is traced back to the seminal work of Haimo in [13], where finite-time stability on second-order (*double integrator*) systems of the form

$$\ddot{x} = u \quad (1)$$

with  $u = u(x, \dot{x})$  continuous, was studied, particularly proving the referred stability property for

$$u = -k_1|x|^a \text{sign}(x) - k_2|\dot{x}|^b \text{sign}(\dot{x}) \quad (2)$$

$k_1 = k_2 = 1$ , with  $b \in (0, 1)$  and  $a > b/(2 - b)$  –or equivalently  $a \in (0, 1)$  and  $b < 2a/(1 + a)$ – [13, Corollary 1], and even stating finite-time stability preservation under (some type of) additional vanishing terms [13, Corollary 2]. Later on, useful foundations were settled down by Bhat and Bernstein [3–7], who stated – for continuous autonomous systems– a formal definition of *finite-time stable equilibrium*, proposed a Lyapunov-based criterion for its

determination, and developed its characterization for *homogeneous* vector fields. This last contribution has been particularly appealing in view of its simplicity since, provided that the origin is an asymptotically stable equilibrium of a homogeneous vector field, finite-time stability is concluded by simply verifying that the degree of homogeneity is negative. Such a simplicity is perceived for instance by comparing the (rather involved) analysis developed in the proof of [13, Corollary 1] against [2, Example 5.6], where finite-time stability on (1)–(2) is analyzed through homogeneity, whence the referred stability property is concluded for  $a \in (0, 1)$  and  $b = 2a/(1 + a)$ , or equivalently  $b \in (0, 1)$  and  $a = b/(2 - b)$ .<sup>1</sup> However, for finite-time control design purposes, such a simple criterion might be restrictive in view of the requirements naturally imposed by homogeneity, which is conventionally a *global* property (see for instance [2] for a formal definition of homogeneous (scalar) functions and vector fields in a coordinate-dependent framework). For instance, in a constrained-input context, the closed-loop system would include bounded components which would preclude the corresponding vector field to be homogeneous [7] (in a coordinate-dependent framework). Nevertheless, such a restriction has been proven to be relaxed through alternative notions of homogeneity [40].

<sup>1</sup> The analyses in [2, Example 5.6] and the proof of [13, Corollary 1] are actually valid for any  $k_1 > 0$  and  $k_2 > 0$ .

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Based on the theoretical framework of local homogeneity [30,40], this work proposes an observer-less output-feedback bounded continuous control scheme for constrained-input mechanical systems, guaranteeing global stabilization with either finite-time or (local) exponential convergence. The choice on the type of stabilization is simply stated through a control parameter. This is made possible through a suitable extension (recently stated in [42]) of the analytical framework of local homogeneity. The finite-time stabilization choice—achieved through bounded (observer-less) output feedback—remains however the main motivation of the present work. This is not only motivated by the implied analytical challenge but also by the advantages of finite-time continuous stabilizers over asymptotic ones—such as faster convergence and improved robustness to uncertainties [15,17,27]—and discontinuous ones [8], as well as their conceptual suitability for certain tasks such as *consensus* [38] or *formation* [39] of multi-agent systems, and process *supervision* (monitoring) [37].

A debuting work on finite-time continuous control for mechanical manipulators was presented in [16] disregarding input constraints. The proposed state-feedback controller adopted Proportional (P) and Derivative (D) type actions with two options on the structure: one of them compensating for the whole system dynamics and the other one only for the gravity terms. The design was based on the conventional analytical framework of homogeneity.

Another work treating the finite-time control of robotic manipulators, assuming unconstrained inputs, appeared later in [43]. The state-feedback scheme proposed therein is designed aiming at the compensation for the whole system nominal dynamics. The rest of the synthesis is developed applying *backstepping*, by viewing the velocity vector as a virtual input to achieve finite-time control of the positions, and relying on the (generalized) force input vector to impose a closed-loop continuous dynamics that guarantees finite-time stabilization of the consequent error variables. The design is then complemented through a *Lyapunov-redesign* type procedure that results in the addition of a control term in charge to reject system uncertainties, which *a priori* renders discontinuous the resulting control law. Alternative approximations of certain control terms are suggested in order to avoid discontinuities and singularities implied by the developed approach, expecting close-enough (to the desired position) stabilization through their replacement.

A different continuous control strategy for the finite-time stabilization of mechanical systems was more recently presented in [36] similarly disregarding input constraints. The proposed state-feedback approach is based on the definition of a (positively invariant) manifold where the system is proven to converge to the zero (desired) state in a finite time  $T_1$ . A suitable closed loop form ensuring convergence of the system variables to such a manifold in a finite time  $T_2$  is then found. The control law is then synthesized through *exact dynamic compensation* so as to impose the closed-loop form found in the precedent step.

Lately, a state-feedback continuous control scheme for the global stabilization with finite-time or (local) exponential convergence of constrained-input mechanical systems was proposed in [42]. It has a generalized saturating PD-type structure involving compensation of the conservative-force terms only. The work includes a simulation study that corroborates the veracity of the so-cited argument claiming that finite-time controllers achieve faster stabilization than asymptotic ones.

From the above-cited state-feedback approaches, only that in [16] formulates an output-feedback extension of the proposed controller (more precisely, of the PD type approach that only involves gravity compensation). It is an observer-based controller that guarantees stabilization only locally, built upon the finite-time observer developed in [14] for the double integrator:

$$\dot{\hat{x}}_1 = \hat{x}_2 + \kappa_1 |x_1 - \hat{x}_1|^c \text{sign}(x_1 - \hat{x}_1) \quad (3a)$$

$$\dot{\hat{x}}_2 = G + \kappa_2 |x_1 - \hat{x}_1|^d \text{sign}(x_1 - \hat{x}_1) \quad (3b)$$

with  $\kappa_1 > 0$ ,  $\kappa_2 > 0$ ,  $d \in (0, 1)$  and  $c = (1 + d)/2$  (or equivalently  $c \in (0.5, 1)$  and  $d = 2c - 1$ ),  $\hat{x}_1$  and  $\hat{x}_2$  being the observer states for the respective reconstruction of  $x_1$  and  $x_2$ , and  $x_1 = x$ ,  $x_2 = \dot{x}$  and  $G = u$  for (1), in [14], while  $x_1$ ,  $x_2$  and  $G$  respectively stand for position, velocity and the respective terms from the dynamic model for every link of the manipulator, in [16]. Thus, the considered finite-time observer involves the whole system dynamics (and parameters), and reconstructs the whole set of system states, i.e. position and velocity variables. Although a bounded variation of such an observer-based output-feedback approach, with the conventional saturation function involved in the P and D type actions, was further contemplated, no formal closed-loop analysis was presented for this case, which does not fit within the analytical framework where the proposed unconstrained schemes were developed (as previously explained).

It is important to keep in mind that by finite-time continuous control on all the above stated discussion and cited references, we mean continuity at every one of the control scheme components, i.e. at the controller output as well as at the auxiliary state equation when dynamic. Efforts to achieve finite-time convergence or stabilization have also been made by involving discontinuities, whether at the controller output or at the auxiliary subsystem (when considered). This is the case for instance of sliding-mode algorithms [23], which aim at leading the system trajectories to a sliding manifold (where the considered stabilization objective is guaranteed) in finite time. These have motivated various finite-time convergent or stabilizing algorithms. For instance, a modified version of a 2nd-order sliding-mode (*super twisting*) algorithm has given rise to a finite-time-convergent differentiator in [24]. Inspired by the super twisting algorithm, a discontinuous version of Hong's finite-time observer has been presented in [11], by proposing the use of Eq. (3) with  $d = 0$  and  $c = 1/2$ ; the observer is addressed to mechanical systems, whence  $G = f(t, x_1, \hat{x}_2, u)$  (in (3b)), such that  $\ddot{x} = f(t, x_1, x_2, u) + \xi(t, x_1, x_2, u)$  is taken to correspond to a generic model of the system dynamics at each one of its degrees of freedom, with  $\xi$  representing uncertainties and  $u$  some control input, under the assumption that the system states (position and velocity variables) can be considered bounded and that, through such an assumption,  $f$  and  $\xi$  in the system dynamics can be considered bounded too. Furthermore, inspired by the twisting algorithm, the finite-time control scheme analyzed in [2] (and previously studied in [13]) for (1), namely (2), has been extended in [31] (under the explicit consideration of control gains  $k_1 > 0$  and  $k_2 > 0$ ) so as to include the discontinuous form adopted by taking  $a = b = 0$  (i.e. stating finite-time stabilization for  $b \in [0, 1)$  and  $a = 2/(2 - b)$ , or equivalently  $a \in [0, 1)$  and  $b = 2a/(1 + a)$ ); this, together with a modified version of the finite-time observer of [11], consisting in the addition of linear position observation error correction terms (in the right-hand side of Eq. (3)), was proposed in [31] as an output-feedback approach for the double integrator; such an output-feedback approach was further applied for the achievement of an orbital stabilization of a bipedal robot under ground unilateral constraints, in [1]. Through the involved discontinuities, at each one of the cited sliding-mode inspired approaches, a higher degree of robustness is earned. The price to pay is a post-transient variation of the system error variables and control signal, although this effect is considerably reduced if the discontinuities are confined to the auxiliary state equation (avoiding discontinuities at the controller output). It is further important to point out that although the discontinuous control approach in [31], obtained by taking  $a = b = 0$  in (2), finishes up by being bounded, input constraints are not considered in the problem formulated in either [31] or [1]; furthermore, the discontinuities implied by such a choice in the controller output would entail the well-known phenomenon of chattering (in view of which such a discontinuous option on the controller output is argued to be avoided in [1]).

Thus, the question on how to succeed the output-feedback finite-time stabilization goal (for mechanical systems) globally, through a fully continuous control scheme (avoiding discontinuities at every one of the controller components: at its output and auxiliary dynamics), and under the consideration of input constraints remains open. This is the question that is analyzed and answered in this study. Moreover, this work aims at solving such an open problem avoiding the use of observers but rather ensuring motion dissipation dynamically from the exclusive feedback of the position variables. This has been made possible in previous output-feedback asymptotic approaches through the use of the so-called *dirty derivative* [19,25,32,35]. By explicitly visualizing its dissipative role [32], in this work, a more general nonlinear version of such a simple operator is designed so as to achieve the required (local) homogeneity properties that permit the finite-time or exponential stabilization. The controller output keeps an SP-SD structure with saturating correction actions on the position error and the output of the generalized dirty-derivative-type subsystem, and although it considers the exclusive compensation for the inherent conservative forces at its output level, it prevents involvement of any one of the terms of the system model in the auxiliary (dirty-derivative-type) state equation. The closed-loop analysis will not only demonstrate the achievement of the formulated global stabilization goal –avoiding input saturation– with *user-predefined* finite-time or (local) exponential convergence, but it also gives a complementary insight on the dissipative role of the generalized dirty-derivative type subsystem through an *ad hoc* feedback-system passivity theorem. A simulation section showing the closed-loop performance achievements through the proposed scheme is included.

## 2. Preliminaries

Let  $X \in \mathbb{R}^{m \times n}$  and  $y \in \mathbb{R}^n$ . Throughout this work,  $X_{ij}$  denotes the element of  $X$  at its  $i$ th row and  $j$ th column,  $X_i$  represents the  $i$ th row of  $X$  and  $y_i$  stands for the  $i$ th element of  $y$ .  $0_n$  represents the origin of  $\mathbb{R}^n$  and  $I_n$  the  $n \times n$  identity matrix. We denote  $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$  and  $\mathbb{R}_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$  for scalars, and  $\mathbb{R}_{>0}^n = \{x \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\}$  and  $\mathbb{R}_{\geq 0}^n = \{x \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\}$  for vectors.  $\|\cdot\|$  stands for the standard Euclidean norm for vectors and induced norm for matrices. An  $(n-1)$ -dimensional sphere of radius  $c > 0$  on  $\mathbb{R}^n$  is denoted  $S_c^{n-1}$ , i.e.  $S_c^{n-1} = \{x \in \mathbb{R}^n : \|x\| = c\}$ . We denote  $\text{sat}(\cdot)$  the standard (unitary) saturation function, i.e.  $\text{sat}(\zeta) = \text{sign}(\zeta) \min\{|\zeta|, 1\}$ . The contents of the following sections – except for Section 2.3 – were mostly included in [42]; for the sake of completeness, they are reproduced here.

### 2.1. Mechanical systems

Consider the  $n$ -DOF fully-actuated frictionless mechanical system dynamics [9, §6.1]

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau \quad (4)$$

where  $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$  are the position (generalized coordinates), velocity, and acceleration vectors,  $H(q) \in \mathbb{R}^{n \times n}$  is the inertia matrix,  $C(q, \dot{q}) \in \mathbb{R}^{n \times n}$  is the Coriolis and centrifugal effect matrix,  $g(q) = \nabla \mathcal{U}(q)$  with  $\mathcal{U} : \mathbb{R}^n \rightarrow \mathbb{R}$  being the potential energy function of the system, and  $\tau \in \mathbb{R}^n$  is the external input (generalized) force vector. Some well-known properties characterizing the terms of such a dynamical model are recalled here [9, §6.1.2] [33, §2.3]. Subsequently, we denote  $\dot{H}$  the rate of change of  $H$ , i.e.  $\dot{H} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  with  $\dot{H}_{ij}(q, \dot{q}) = \frac{\partial H_{ij}}{\partial \dot{q}}(q)\dot{q}$ ,  $i, j = 1, \dots, n$ .

**Property 1.**  $H(q)$  is a continuously differentiable positive definite symmetric matrix function.

**Property 2.** The Coriolis and centrifugal effect matrix satisfies:

$$\begin{aligned} 2.1 \quad & \dot{q}^T \left[ \frac{1}{2} \dot{H}(q, \dot{q}) - C(q, \dot{q}) \right] \dot{q} = 0, \quad \forall (q, \dot{q}) \in \mathbb{R}^n \times \mathbb{R}^n; \\ 2.2 \quad & C(x, y)z = C(x, z)y, \quad \forall x, y, z \in \mathbb{R}^n. \end{aligned}$$

**Remark 1.** Observe from Property 2.2 that  $C(q, a\dot{q})b\dot{q} = C(q, b\dot{q})a\dot{q} = C(q, ab\dot{q})\dot{q} = C(q, \dot{q})ab\dot{q}$ ,  $\forall q, \dot{q} \in \mathbb{R}^n, \forall a, b \in \mathbb{R}$ .

In this work, we consider the (realistic) bounded input case, where the absolute value of each input  $\tau_i$  is constrained to be smaller than a given saturation bound  $T_i > 0$ , i.e.  $|\tau_i| \leq T_i$ ,  $i = 1, \dots, n$ . More precisely, letting  $u_i$  represent the control variable (controller output) relative to the  $i$ th degree of freedom, we have that

$$\tau_i = T_i \text{sat}(u_i/T_i) \quad (5)$$

Further assumptions are stated next.

**Assumption 1.** The conservative (generalized) force vector  $g(q)$  is bounded, or equivalently, every one of its elements,  $g_i(q)$ ,  $i = 1, \dots, n$ , satisfies  $|g_i(q)| \leq B_{gi}$ ,  $\forall q \in \mathbb{R}^n$ , for some positive constant  $B_{gi}$ .

**Assumption 2.**  $T_i > B_{gi}$ ,  $\forall i \in \{1, \dots, n\}$ .

Assumption 1 applies e.g. for robot manipulators having only revolute joints [20, §4.3]. Assumption 2 renders it possible to hold the system at any desired equilibrium configuration  $q_d \in \mathbb{R}^n$ .

### 2.2. Local homogeneity, finite-time stability and $\delta$ -exponential stability

The following homogeneity-related definitions are stated under the consideration of coordinates  $(x_1, \dots, x_n)$  in  $\mathbb{R}^n$ . We begin by introducing the notion of *family of dilations*  $\delta_\varepsilon^r$ , defined as  $\delta_\varepsilon^r(x) = (\varepsilon^{r_1}x_1, \dots, \varepsilon^{r_n}x_n)^T$ ,  $\forall x = (x_1, \dots, x_n)^T \in \mathbb{R}^n$ ,  $\forall \varepsilon > 0$ , with  $r = (r_1, \dots, r_n)^T$ , where the *dilation coefficients*  $r_1, \dots, r_n$  are positive scalars. Fundamental in this study is the concept of *local homogeneity*, notions of which are stated –in a coordinate-dependent framework– in [30], under the explicit consideration of time (in addition to coordinates), and in [40] in the time-invariant case.

**Definition 1.** A function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , resp. vector field  $f = \sum_{i=1}^n f_i \frac{\partial}{\partial x_i}$  (with  $f_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ), is *locally homogeneous of degree  $\alpha$*  with respect to the family of dilations  $\delta_\varepsilon^r$  –or equivalently, it is said to be *locally  $r$ -homogeneous of degree  $\alpha$* – if there exists an open neighborhood of the origin  $D \subset \mathbb{R}^n$  –referred to as the *domain of homogeneity*– such that, for every  $x \in D$  and all  $\varepsilon \in (0, 1]$ :  $\delta_\varepsilon^r(x) \in D$  and

$$V(\delta_\varepsilon^r(x)) = \varepsilon^\alpha V(x) \quad (6)$$

resp.

$$f_i(\delta_\varepsilon^r(x)) = \varepsilon^{\alpha+r_i} f_i(x) \quad (7)$$

$i = 1, \dots, n$ .

Let us note that an  $r$ -homogeneous (in the conventional sense) function, resp. vector field, is a locally  $r$ -homogeneous function, resp. vector field, with domain of homogeneity  $D = \mathbb{R}^n$ .

**Definition 2.** [28] Given  $r \in \mathbb{R}_{>0}^n$ , a continuous function mapping  $x \in \mathbb{R}^n$  to  $\mathbb{R}$ , denoted  $\|x\|_r$ , is called a *homogeneous norm* with respect to the family of dilations  $\delta_\varepsilon^r$  –or equivalently, it is said to be an  *$r$ -homogeneous norm*– if  $\|x\|_r \geq 0$  with  $\|x\|_r = 0 \iff x = 0_n$ , and  $\|\delta_\varepsilon^r(x)\|_r = \varepsilon \|x\|_r$  for any  $\varepsilon > 0$ .

Notice that an  $r$ -homogeneous norm is a positive definite continuous function being  $r$ -homogeneous of degree 1.

**Definition 3.** An  $r$ -homogeneous  $(n-1)$ -sphere of radius  $c > 0$  is the set  $S_{r,c}^{n-1} = \{x \in \mathbb{R}^n : \|x\|_r = c\}$ .

A special subset of  $r$ -homogeneous norms is defined as follows.

**Definition 4.** [18] Given  $r \in \mathbb{R}_{>0}^n$ , an  $r$ -homogeneous  $p$ -norm ( $p \geq 1$ ) is defined as  $\|x\|_{r,p} = [\sum_{i=1}^n |x_i|^{p/r_i}]^{1/p}$ .

Subsequently, in this work, an  $r$ -homogeneous norm  $\|\cdot\|_r$  will be considered to refer to an  $r$ -homogeneous  $p$ -norm with  $p > \max_i \{r_i\}$ .

Consider an  $n$ th order autonomous system

$$\dot{x} = f(x) \tag{8}$$

where  $f : \mathcal{D} \rightarrow \mathbb{R}^n$  is continuous on an open neighborhood of the origin  $\mathcal{D} \subset \mathbb{R}^n$  and  $f(0_n) = 0_n$ , and let  $x(t; x_0)$  represent the system solution with initial condition  $x(0; x_0) = x_0$ .

**Definition 5.** [3,7] The origin is said to be a *finite-time stable* equilibrium of system (8) if it is Lyapunov stable and there exist an open neighborhood of the origin,  $\mathcal{N} \subset \mathcal{D}$ , being positively invariant with respect to (8), and a positive definite function  $T : \mathcal{N} \rightarrow \mathbb{R}_{>0}$ , called the *settling-time function*, such that  $x(t; x_0) \neq 0_n, \forall t \in [0, T(x_0)), \forall x_0 \in \mathcal{N} \setminus \{0_n\}$ , and  $x(t; x_0) = 0_n, \forall t \geq T(x_0), \forall x_0 \in \mathcal{N}$ . The origin is said to be a *globally finite-time stable* equilibrium if it is finite-time stable with  $\mathcal{N} = \mathcal{D} = \mathbb{R}^n$ .

**Remark 2.** Note, from Definition 5, that the origin is a globally finite-time stable equilibrium of system (8) if and only if it is globally asymptotically stable and finite-time stable.

The next theorem, stating a local-homogeneity-based necessary-and-sufficient criterion for global finite-time stability, is reproduced from [40]. A previous version stating the sufficiency part of the theorem and providing an upper estimate of the settling time is found in [30].

**Theorem 1.** Consider system (8) with  $\mathcal{D} = \mathbb{R}^n$ . Suppose that  $f$  is a locally  $r$ -homogeneous vector field of degree  $\alpha$  with domain of homogeneity  $D \subset \mathbb{R}^n$ . Then, the origin is a globally finite-time stable equilibrium of system (8) if and only if it is globally asymptotically stable and  $\alpha < 0$ .

The next definition is stated under the additional consideration that, for some  $r \in \mathbb{R}_{>0}^n$ ,  $f$  in (8) is locally  $r$ -homogeneous with domain of homogeneity  $D \subset \mathcal{D}$ .

**Definition 6.** [18,28] The equilibrium point  $x = 0_n$  of (8) is  $\delta$ -exponentially stable<sup>2</sup> with respect to the homogeneous norm  $\|\cdot\|_r$  if there exist a neighborhood of the origin,  $\mathcal{V} \subset D$ , and constants  $a \geq 1$  and  $b > 0$  such that  $\|x(t; x_0)\|_r \leq a\|x_0\|_r e^{-bt}, \forall t \geq 0, \forall x_0 \in \mathcal{V}$ .

**Remark 3.** Observe that Definition 6 becomes equivalent to the usual definition of exponential stability when the standard dilation is concerned, i.e. when  $r_i = 1, i = 1, \dots, n$ .

The next lemma is a trivial extension to the local homogeneity context of [18, Lemma 2.4]. Analogously to [18, Lemma 2.4], it is stated under the additional consideration that solutions of (8) with  $x_0 \in D$  remain unique (while belonging to  $D$ ).<sup>3</sup>

**Lemma 1.** Suppose that  $f$  in (8) is a locally  $r$ -homogeneous vector field of degree  $\alpha = 0$  with domain of homogeneity  $D \subset \mathcal{D}$ . Then, the origin is a  $\delta$ -exponentially stable equilibrium if and only if it is asymptotically stable.

<sup>2</sup> We adopt the dilation-related designation stated in [18] for Definition 6, i.e.  $\delta$ -exponential stability. In [28], the same definition is alternatively designated as  $\rho$ -exponential stability, with  $\rho$  referring to the involved  $r$ -homogeneous norm, in accordance to the notation stated therein.

<sup>3</sup> Another version of [18, Lemma 2.4] is stated in [28, Lemma 1] where no restriction on the uniqueness of solutions is considered. It is further concluded from [28, §III.E] that the solutions of autonomous systems  $\dot{x} = f(x)$  with  $r$ -homogeneous vector field being locally Lipschitz on  $\mathbb{R}^n \setminus \{0_n\}$  are unique.

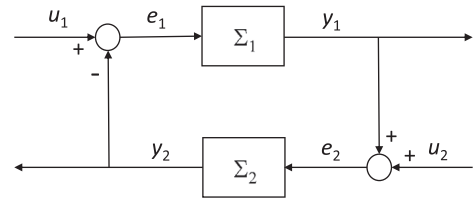


Fig. 1. Feedback connection.

Observe that the assumptions of Lemma 1 imply the existence of a neighborhood of the origin  $\mathcal{V} \subset D$  such that  $x_0 \in \mathcal{V} \Rightarrow x(t; x_0) \in D, \forall t \geq 0$ . The proof of Lemma 1 is thus analogous to the one developed in [12, §57] for the special case of  $r = (r_1, \dots, r_n)^T$  with  $r_i = 1, i = 1, \dots, n$ .<sup>4</sup>

**Remark 4.** Let us note that if a vector field  $f$  is locally  $r$ -homogeneous of degree  $\alpha = 0$  with dilation coefficients  $r_i = r_0, \forall i \in \{1, \dots, n\}$ , for some  $r_0 > 0$ , then  $f$  is locally  $r^*$ -homogeneous of degree  $\alpha = 0$  with dilation coefficients  $r_i^* = r_0^*, \forall i \in \{1, \dots, n\}$ , for any  $r_0^* > 0$ . Indeed, observe that if, for every  $x \in D, f(\varepsilon^{r_0} x) = \varepsilon^{r_0} f(x), \forall \varepsilon \in (0, 1]$ , then, by taking  $\varepsilon = \varepsilon^{r_0/r_0^*}$ , we have that  $f(\varepsilon^{r_0^*} x) = \varepsilon^{r_0^*} f(x), \forall \varepsilon \in (0, 1]$ . Consequently, if  $f$  in (8) is locally  $r$ -homogeneous of degree  $\alpha = 0$  with dilation coefficients  $r_i = r_0, \forall i \in \{1, \dots, n\}$ , for some  $r_0 > 0$ , then (under the consideration of Remark 3) the origin turns out to be exponentially stable if and only if it is  $\delta$ -exponentially stable.

Consider an  $n$ th order autonomous system of the form

$$\dot{x} = f(x) + \hat{f}(x) \tag{9}$$

where  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $\hat{f} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  are continuous vector fields such that  $f(0_n) = \hat{f}(0_n) = 0_n$ . The next result is an extended version of [40, Lemma 3.2].

**Lemma 2.** [42] Suppose that, for some  $r \in \mathbb{R}_{>0}^n, f$  in (9) is a locally  $r$ -homogeneous vector field of degree  $\alpha < 0$ , resp.  $\alpha = 0$ , with domain of homogeneity  $D \subset \mathbb{R}^n$ , and that  $0_n$  is a globally asymptotically, resp.  $\delta$ -exponentially, stable equilibrium of  $\dot{x} = f(x)$ . Then, the origin is a finite-time, resp.  $\delta$ -exponentially, stable equilibrium of system (9) if

$$\lim_{\varepsilon \rightarrow 0^+} \frac{\hat{f}_i(\delta_\varepsilon^r(x))}{\varepsilon^{\alpha+r_i}} = 0 \tag{10}$$

$i = 1, \dots, n, \forall x \in S_{r,c}^{n-1}, \text{ resp. } \forall x \in S_{r,c}^{n-1}, \text{ for some } c > 0 \text{ such that } S_{r,c}^{n-1} \subset D, \text{ resp. } S_{r,c}^{n-1} \subset D.$

**Remark 5.** Notice that the condition required by Lemma 2 may be equivalently verified through the satisfaction of

$$\lim_{\varepsilon \rightarrow 0^+} \|\varepsilon^{-\alpha} \text{diag}[\varepsilon^{-r_1}, \dots, \varepsilon^{-r_n}] \hat{f}(\delta_\varepsilon^r(x))\| = 0 \tag{11}$$

$\forall x \in S_{r,c}^{n-1}$  (resp.  $S_{r,c}^{n-1}$ ). In other words, (10) is fulfilled for all  $i = 1, \dots, n$  and all  $x \in S_{r,c}^{n-1}$  (resp.  $S_{r,c}^{n-1}$ ) if and only if (11) is satisfied for all  $x \in S_{r,c}^{n-1}$  (resp.  $S_{r,c}^{n-1}$ ).

### 2.3. Passivity

Basic definitions are recalled in Appendix A. Consider here the feedback system of Fig. 1, where each feedback component  $\Sigma_i, i = 1, 2$ , is represented by the state model

$$\dot{x}_i = f_i(x_i, e_i) \tag{12a}$$

$$y_i = h_i(x_i, e_i) \tag{12b}$$

<sup>4</sup> One further concludes from [12, §57] that asymptotic stability when  $\alpha > 0$  is not  $\delta$ -exponential (i.e.  $\delta$ -exponential stability is a property that can only take place when  $\alpha = 0$ ).

with  $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \rightarrow \mathbb{R}^{n_i}$  and  $h_i : \mathbb{R}^{n_i} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  being continuous,  $f_i(x_i, e_i)$  locally Lipschitz on  $\mathbb{R}^{n_i} \times \mathbb{R}^m \setminus (0_{n_i}, 0_m)$ ,  $f_i(0_{n_i}, 0_m) = 0_{n_i}$  and  $h_i(0_{n_i}, 0_m) = 0_m$ . We will consider that the feedback connection is well-defined<sup>5</sup> [21, §6.5]. We state the following *feedback-system passivity theorem*.

**Theorem 2.** For the considered feedback connection with  $u_1 = u_2 = 0_m$ , the origin of the consequent closed-loop system,  $(x_1, x_2) = (0_{n_1}, 0_{n_2})$ , is asymptotically stable if, for some  $i \in \{1, 2\}$  and  $j \in \{1, 2\} \setminus \{i\}$  (or equivalently  $j = i - (-1)^i$ ),  $\Sigma_i$  is zero-state observable and passive with positive definite storage function,  $\Sigma_j$  is strictly passive and

$$f_j(0_{n_j}, e_j) = 0_{n_j} \Rightarrow e_j = 0_m \quad (13)$$

Furthermore, if the storage function for each component is radially unbounded, the origin is globally asymptotically stable.

**Proof.** See Appendix B.  $\square$

Theorem 2 keeps a close relation with previous passivity theorems or passivity-related results for feedback systems. Indeed, it is well known that an unforced feedback connection –as depicted in Fig. 1– where one of the components,  $\Sigma_j$ , is a dynamical system being strictly passive, and the other component,  $\Sigma_i$ , is a passive function (memoryless process), yields the origin of the closed loop (globally) asymptotically stable [21, Theorem 6.4],<sup>6</sup> as it is also the case when the dynamical component is passive and zero-state observable –or even zero-state detectable– and the static one is a passive function whose internal product with any non-zero (vector) value of its argument is (strictly) positive [10]. Theorem 2 goes in an analog direction of such well-known results, but under the consideration of both components being dynamical systems. A similar approach has been previously used, for instance, on the design of output-feedback schemes for Lagrangian systems [26,32], by considering their feedback connection with a controller that keeps an Euler-Lagrange structure too, basing such a design philosophy on the natural passivity properties of such type of systems. In this direction, it is worth noting that Theorem 2 is not addressed to interconnected systems of particular structure, and that, in such a general context, it clearly brings to the fore the additional requirement (condition (13)) that makes the stated result possible.

#### 2.4. Scalar functions with particular properties

**Definition 7.** A continuous scalar function  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$  will be said to be:

1. *bounded* (by  $M$ ) if  $|\sigma(\zeta)| \leq M$ ,  $\forall \zeta \in \mathbb{R}$ , for some positive constant  $M$ ;
2. *strictly passive*<sup>7</sup> if  $\zeta\sigma(\zeta) > 0$ ,  $\forall \zeta \neq 0$ ;
3. *strongly passive* if it is a strictly passive function satisfying  $|\sigma(\zeta)| \geq \kappa |a \operatorname{sat}(\zeta/a)|^b = \kappa (\min\{|\zeta|, a\})^b$ ,  $\forall \zeta \in \mathbb{R}$ , for some positive constants  $\kappa$ ,  $a$  and  $b$ .

**Remark 6.** Notice that equivalent characterizations of strictly passive functions are:  $\zeta\sigma(\zeta) > 0 \iff \operatorname{sign}(\zeta)\sigma(\zeta) > 0 \iff \operatorname{sign}(\sigma(\zeta)) = \operatorname{sign}(\zeta)$ ,  $\forall \zeta \neq 0$ .

Let us note that a non-decreasing strictly passive function  $\sigma$  is strongly passive. Indeed, notice that the strictly passive character of  $\sigma$  implies the existence of a sufficiently small  $a > 0$  such

<sup>5</sup> Independence of  $h_i$  on  $e_i$  for either  $i \in \{1, 2\}$  suffices to ensure that the feedback connection is well-defined.

<sup>6</sup> See also [21, Theorem 6.5] where *shortage* of passivity is permitted in either of the components as long as a dominating *excess* of passivity characterizes the other component.

<sup>7</sup> The designation stated in items 2 and 3 of Definition 7 is inspired on the definition of a passive memoryless process [21, §6.1].

that  $|\sigma(\zeta)| \geq \kappa |\zeta|^b$ ,  $\forall |\zeta| \leq a$ , for some positive constants  $\kappa$  and  $b$ , while from its nondecreasing character we have that  $|\sigma(\zeta)| \geq |\sigma(\operatorname{sign}(\zeta)a)| \geq \kappa a^b$ ,  $\forall |\zeta| \geq a$ , and thus  $|\sigma(\zeta)| \geq \kappa (\min\{|\zeta|, a\})^b = \kappa |a \operatorname{sat}(\zeta/a)|^b$ ,  $\forall \zeta \in \mathbb{R}$ .

**Lemma 3.** [42] Let  $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ ,  $\sigma_0 : \mathbb{R} \rightarrow \mathbb{R}$  and  $\sigma_1 : \mathbb{R} \rightarrow \mathbb{R}$  be strongly passive functions and  $k$  be a positive constant. Then:

1.  $\int_0^\zeta \sigma(kv)dv > 0$ ,  $\forall \zeta \neq 0$ ;
2.  $\int_0^\zeta \sigma(kv)dv \rightarrow \infty$  as  $|\zeta| \rightarrow \infty$ ;
3.  $\sigma_0 \circ \sigma_1$  is strongly passive.

### 3. The proposed output-feedback scheme

Consider the following SP-SD type controller

$$u(q, \vartheta) = -s_1(K_1 \bar{q}) - s_2(K_2 \vartheta) + g(q) \quad (14)$$

where  $\bar{q} = q - q_d$ , for any constant (desired equilibrium position)  $q_d \in \mathbb{R}^n$ ;  $\vartheta \in \mathbb{R}^n$  is the output vector variable of an auxiliary subsystem defined as

$$\dot{\vartheta}_c = -As_3(\vartheta_c + B\bar{q}) \quad (15a)$$

$$\vartheta = \vartheta_c + B\bar{q} \quad (15b)$$

$K_1$ ,  $K_2$ ,  $A$  and  $B$  are positive definite diagonal matrices, i.e.  $K_i = \operatorname{diag}[k_{i1}, \dots, k_{in}]$ ,  $i = 1, 2$ ,  $A = \operatorname{diag}[a_1, \dots, a_n]$  and  $B = \operatorname{diag}[b_1, \dots, b_n]$ , with  $k_{ij} > 0$ ,  $a_j > 0$  and  $b_j > 0$ ,  $\forall j \in \{1, \dots, n\}$ ; and for any  $x \in \mathbb{R}^n$ ,  $s_i(x) = (\sigma_{i1}(x_1), \dots, \sigma_{in}(x_n))^T$ ,  $i = 1, 2, 3$ , with, for each  $j \in \{1, \dots, n\}$ ,  $\sigma_{3j}(\cdot)$  being a strictly passive function, while  $\sigma_{1j}$  and  $\sigma_{2j}$  are strongly passive functions such that<sup>8</sup>

$$B_j \triangleq \max_{(\zeta_1, \zeta_2) \in \mathbb{R}^2} |\sigma_{1j}(\zeta_1) + \sigma_{2j}(\zeta_2)| < T_j - B_{gj} \quad (16)$$

all three being locally Lipschitz-continuous on  $\mathbb{R} \setminus \{0\}$ .

**Remark 7.** Note that, by (16), we have that –for each  $j \in \{1, \dots, n\}$ –  $\sigma_{1j}$  and  $\sigma_{2j}$  shall both be bounded, while  $\sigma_{3j}$  may be bounded or not. Moreover, the bounds of  $\sigma_{1j}$  and  $\sigma_{2j}$  are naturally restricted by (16), while an eventual choice of a bounded  $\sigma_{3j}$  would permit a free selection on its bound.

**Remark 8.** Let us note that the auxiliary subsystem in Eq. (15) is a nonlinear version of the *dirty derivative* operator, applied to the position error vector variable. Indeed, observe that if  $s_3$  were chosen to be the identity function, i.e.  $s_3(x) \equiv x$ , the conventional linear dynamics of the dirty derivative (applied to  $\bar{q}$ ) [32] is obtained. Through the required analytical properties on  $s_3$ , the expected closed-loop stability features will be proven to be obtained. Further requirements on  $s_3$  will show the usefulness of such a generalized form to get the focused types of trajectory convergence. The output variable of the (non-linear) dirty-derivative-type subsystem,  $\vartheta$ , may thus be seen as an approximated dirty derivative of  $\bar{q}$  –or an approximated *dirty* calculation of  $\dot{\bar{q}}$ – even though a more appropriate insight on the role played by the auxiliary subsystem in Eq. (15) will be brought to the fore (later, in Remark 10) under an energy-related optics.

**Remark 9.** The control scheme in Eqs. (14) and (15) is reminiscent of that in [26], which was shown to be obtained through the design methodology proposed therein for the asymptotic stabilization of Lagrangian systems under input constraints, and includes the *desired-conservative-force* pre-compensation option.

<sup>8</sup> Notice that if  $\sigma_{1j}$  and  $\sigma_{2j}$  are (both) chosen to be non-decreasing, then  $B_j = \max \left\{ \lim_{\zeta \rightarrow \infty} [\sigma_{1j}(\zeta) + \sigma_{2j}(\zeta)], \lim_{\zeta \rightarrow -\infty} -[\sigma_{1j}(\zeta) + \sigma_{2j}(\zeta)] \right\}$ .

Such a scheme in [26] has been among the first controllers of its kind and inspired similar alternative bounded-input control design formulations for Euler–Lagrange type systems. Leaving aside the conservative-force compensation aspect, the control structure proposed through Eqs. (14) and (15) in this work goes however further in its generalization by permitting the choice among finite-time or exponential stabilization, as will be stated and proven next, and by allowing larger design flexibility on the functions  $\sigma_{ij}$ ,  $i = 1, 2, 3$ ,  $j = 1, \dots, n$ , involved to guarantee the formulated control objective. It is worth adding that the desired pre-compensation option could also be included here through additional requirements ensuring that the potential energy component due to the first term in the right-hand side of (14) dominates those of the conservative-force and desired pre-compensation terms of the closed loop. Details on this option will be reported on future communications.

**Proposition 1.** Consider system (4) and (5) in closed loop with the proposed control scheme in Eqs. (14) and (15). Thus, for any positive definite diagonal matrices  $K_1$ ,  $K_2$ ,  $A$  and  $B$ : global asymptotic stability of the closed-loop trivial solution  $\bar{q}(t) \equiv 0_n$  is guaranteed with  $|\tau_j(t)| = |u_j(t)| < T_j$ ,  $j = 1, \dots, n$ ,  $\forall t \geq 0$ .

**Proof.** Observe that –for every  $j \in \{1, \dots, n\}$ – by (16), we have that, for any  $(q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$ :

$$\begin{aligned} |u_j(q, \vartheta)| &= \left| -\sigma_{1j}(k_{1j}\bar{q}_j) - \sigma_{2j}(k_{2j}\vartheta_j) + g_j(q) \right| \\ &\leq \left| \sigma_{1j}(k_{1j}\bar{q}_j) + \sigma_{2j}(k_{2j}\vartheta_j) \right| + |g_j(q)| \\ &\leq B_j + B_{g_j} < T_j \end{aligned}$$

From this and (5), one sees that  $T_j > |u_j(q, \vartheta)| = |u_j| = |\tau_j|$ ,  $\forall (q, \vartheta) \in \mathbb{R}^n \times \mathbb{R}^n$ , which shows that, along the system trajectories,  $|\tau_j(t)| = |u_j(t)| < T_j$ ,  $j = 1, \dots, n$ ,  $\forall t \geq 0$ . This proves that, under the proposed scheme, the input saturation values,  $T_j$ , are never reached. Hence, the closed-loop dynamics takes the (equivalent) form

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} = -s_1(K_1\bar{q}) - s_2(K_2\vartheta)$$

$$\dot{\vartheta} = -As_3(\vartheta) + B\dot{q}$$

By defining  $x_1 = \bar{q}$ ,  $x_2 = \dot{q}$  and  $x_3 = \vartheta$ , the closed-loop dynamics adopts the 3n-order state-space representation

$$\dot{x}_1 = x_2 \quad (17a)$$

$$\dot{x}_2 = -H^{-1}(x_1 + q_d) \left[ C(x_1 + q_d, x_2)x_2 + s_1(K_1x_1) + s_2(K_2x_3) \right] \quad (17b)$$

$$\dot{x}_3 = -As_3(x_3) + Bx_2 \quad (17c)$$

By further defining  $x = (x_1^T, x_2^T, x_3^T)^T$ , these state equations may be rewritten in the form of system (9) with

$$f(x) = \begin{pmatrix} x_2 \\ -H^{-1}(q_d) \left[ s_1(K_1x_1) + s_2(K_2x_3) \right] \\ -As_3(x_3) + Bx_2 \end{pmatrix} \quad (18a)$$

$$\hat{f}(x) = \begin{pmatrix} 0_n \\ -H^{-1}(x_1 + q_d)C(x_1 + q_d, x_2)x_2 - \mathcal{H}(x_1) \left[ s_1(K_1x_1) + s_2(K_2x_3) \right] \\ 0_n \end{pmatrix} \quad (18b)$$

where

$$\mathcal{H}(x_1) = H^{-1}(x_1 + q_d) - H^{-1}(q_d) \quad (19)$$

Thus, the closed-loop stability property stated through Proposition 1 is corroborated by showing that  $x = 0_{3n}$  is a globally asymptotically stable equilibrium of the state equation  $\dot{x} = f(x) + \hat{f}(x)$ , which is proven through the following theorem (whose formulation proves to be convenient for subsequent developments and proofs).  $\square$

**Theorem 3.** Under the stated specifications, the origin is a globally asymptotically stable equilibrium of both the state equation  $\dot{x} = f(x)$  and the (closed-loop) system  $\dot{x} = f(x) + \hat{f}(x)$ , with  $f(x)$  and  $\hat{f}(x)$  defined through Eq. (18).

**Proof.** For every  $\ell \in \{0, 1\}$ , let us define the continuously differentiable scalar function

$$\begin{aligned} V_\ell(x_1, x_2, x_3) &= \frac{1}{2}x_2^T H(\ell x_1 + q_d)x_2 + \int_{0_n}^{x_1} s_1^T(K_1r)dr \\ &\quad + \int_{0_n}^{x_3} s_2^T(K_2r)B^{-1}dr \end{aligned}$$

where

$$\begin{aligned} \int_{0_n}^{x_1} s_1^T(K_1r)dr &= \sum_{j=1}^n \int_0^{x_{1j}} \sigma_{1j}(k_{1j}r_j)dr_j \\ \int_{0_n}^{x_3} s_2^T(K_2r)B^{-1}dr &= \sum_{j=1}^n \int_0^{x_{3j}} \frac{\sigma_{2j}(k_{2j}r_j)}{b_j} dr_j \end{aligned}$$

From Property 1 and Lemma 3,  $V_\ell(x_1, x_2, x_3)$ ,  $\ell = 0, 1$ , are concluded to be positive definite and radially unbounded. Further, for every  $\ell \in \{0, 1\}$ , the derivative of  $V_\ell$  along the trajectories of  $\dot{x} = f(x) + \ell\hat{f}(x)$ , is obtained as

$$\begin{aligned} \dot{V}_\ell(x_1, x_2, x_3) &= x_2^T H(\ell x_1 + q_d)\dot{x}_2 + \frac{\ell}{2}x_2^T \dot{H}(x_1 + q_d, x_2)x_2 + s_1^T(K_1x_1)\dot{x}_1 \\ &\quad + s_2^T(K_2x_3)B^{-1}\dot{x}_3 \\ &= x_2^T \left[ -\ell C(x_1 + q_d, x_2)x_2 - s_1(K_1x_1) - s_2(K_2x_3) \right] \\ &\quad + \frac{\ell}{2}x_2^T \dot{H}(x_1 + q_d, x_2)x_2 + s_1^T(K_1x_1)x_2 \\ &\quad + s_2^T(K_2x_3)B^{-1} \left[ -As_3(x_3) + Bx_2 \right] \\ &= -s_2^T(K_2x_3)B^{-1}As_3(x_3) \\ &= -\sum_{j=1}^n \frac{a_j}{b_j} \sigma_{2j}(k_{2j}x_{3j})\sigma_{3j}(x_{3j}) \end{aligned}$$

where, in the case of  $\ell = 1$ , Property 2.1 has been applied. Note, from the strictly passive character of  $\sigma_{2j}$  and  $\sigma_{3j}$  (recall Definition 7 and Remark 6),  $j = 1, \dots, n$ , that  $\dot{V}_\ell(x_1, x_2, x_3) \leq 0$ ,  $\forall (x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ , with  $Z_\ell \triangleq \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \dot{V}_\ell(x_1, x_2, x_3) = 0\} = \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : x_3 = 0_n\}$ . Further, from the system dynamics  $\dot{x} = f(x) + \ell\hat{f}(x)$  –under the consideration of the strictly passive character of  $\sigma_{1j}$ ,  $j = 1, \dots, n$ , Property 1 and the positive definiteness of  $K_1$ – one sees that  $x_3(t) \equiv 0_n \Rightarrow \dot{x}_3(t) \equiv 0_n \Rightarrow x_2(t) \equiv 0_n \Rightarrow \dot{x}_2(t) \equiv 0_n \Rightarrow s_1(K_1x_1(t)) \equiv 0_n \iff x_1(t) \equiv 0_n$  (which shows that  $(x_1, x_2, x_3)(t) \equiv (0_n, 0_n, 0_n)$  is the only system solution completely remaining in  $Z_\ell$ ), and corroborates that at any  $(x_1, x_2, x_3) \in Z_\ell \setminus \{(0_n, 0_n, 0_n)\}$ , the resulting unbalanced force terms act on the closed-loop dynamics  $[\dot{x} = f(x_1, x_2, 0_n) + \ell\hat{f}(x_1, x_2, 0_n)]$  with  $(x_1, x_2) \neq (0_n, 0_n)$ , forcing the system trajectories to leave  $Z_\ell$ , whence  $\{(0_n, 0_n, 0_n)\}$  is concluded to be the only invariant set in  $Z_\ell$ ,  $\ell = 0, 1$ . Therefore, by the invariance theory [29, §7.2] (more precisely by [29, Corollary 7.2.1]<sup>9</sup>),  $x = 0_{3n}$  is concluded to be a globally asymptotically stable equilibrium of both the state equation  $\dot{x} = f(x)$  and the (closed-loop) system  $\dot{x} = f(x) + \hat{f}(x)$ .  $\square$

<sup>9</sup> Corollary 7.2.1 in [29] is a version of Barbashin–Krasovskii's theorem that permits to conclude on the global asymptotic stability of the origin of an autonomous system  $\dot{x} = f(x)$  under the consideration of a continuous vector field  $f(x)$ . This is in contrast to other well-known versions like [21, Corollaries 4.1 & 4.2], that require  $f(x)$  to be locally Lipschitz, or those in [22,34] where  $f(x)$  is considered to be continuously differentiable.

**Remark 10.** Consider the closed-loop system in Eq. (17). Let  $e_1 = -y_2 = -s_2(K_2x_3)$ ,  $e_2 = y_1 = x_2$ ,  $\psi(x_3) = s_2^T(K_2x_3)B^{-1}As_3(x_3)$ ,

$$V_{11}(x_1, x_2) = \frac{1}{2}x_2^T H(x_1 + q_d)x_2 + \int_{0_n}^{x_1} s_1^T(K_1r)dr$$

and

$$V_{12}(x_3) = \int_{0_n}^{x_3} s_2^T(K_2r)B^{-1}dr$$

By previous arguments and developments,  $V_{11}$  and  $V_{12}$  are radially unbounded positive definite functions in their respective arguments. Following an analysis analog to that of the proof of Theorem 3, one obtains

$$\dot{V}_{11} = e_1^T y_1$$

and

$$\dot{V}_{12} = e_2^T y_2 - \psi(x_3)$$

with  $\psi(x_3)$  being positive definite (in its argument). Hence, the closed-loop system in Eq. (17) may be seen as a (negative) feedback system connection—as depicted in Fig. 1—among a passive—actually lossless—subsystem  $\Sigma_1$  with dynamic model

$$\Sigma_1 : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = H^{-1}(x_1 + q_d) [-C(x_1 + q_d, x_2)x_2 - s_1(K_1x_1) + e_1] \\ y_1 = x_2 \end{cases}$$

and positive definite storage function  $V_{11}(x_1, x_2)$ , and a strictly passive subsystem  $\Sigma_2$  with state model

$$\Sigma_2 : \begin{cases} \dot{x}_3 = -As_3(x_3) + Be_2 \triangleq f_2(x_3, e_2) \\ y_2 = s_2(K_2x_3) \end{cases} \quad (20)$$

and storage function  $V_{12}(x_3)$ . Moreover, one sees from (20) that  $f_2(0_n, e_2) = Be_2 = 0_n \Rightarrow e_2 = 0_n$ , completing the requirements of Theorem 2. This formulation actually brings to the fore the damping-injection role that subsystem (17c)—or, equivalently, in Eq. (15) (in the original coordinates)—plays in the closed loop. Indeed, through its  $x_3$ -dependent term, subsystem (17c) in fact acts as a dynamic damper in charge to dissipate the feedback system stored energy, thus leading the closed-loop trajectories to the (unique) minimum-energy configuration, located (by feedback) at the desired position.

#### 4. Finite-time and exponential stabilization

**Proposition 2.** Consider the proposed control scheme under the additional consideration that, for every  $j \in \{1, \dots, n\}$ ,  $\sigma_{ij}$ ,  $i = 1, 2$ , are locally  $r_i$ -homogeneous of (common) degree  $\alpha_i = 2r_2 - r_1 > 0$ —i.e.  $r_{1j} = r_1$ ,  $r_{2j} = r_2$  and  $\alpha_{1j} = \alpha_1 = 2r_2 - r_1 = \alpha_2 = \alpha_{2j} > 0$  for all  $j \in \{1, \dots, n\}$ —with domain of homogeneity  $D_{ij} = \{\zeta \in \mathbb{R} : |\zeta| < L_{ij} \in (0, \infty)\}$  and  $\sigma_{3j}$  is locally  $r_1$ -homogeneous of degree  $\alpha_3 = r_2$ —i.e.  $r_{3j} = r_3 = r_1$  and  $\alpha_{3j} = \alpha_3 = r_2$  for all  $j \in \{1, \dots, n\}$ —with domain of homogeneity  $D_{3j} = \{\zeta \in \mathbb{R} : |\zeta| < L_{3j} \in (0, \infty)\}$ . Thus, for any positive definite diagonal matrices  $K_1, K_2, A$  and  $B$ :  $|\tau_j(t)| = |u_j(t)| < T_j$ ,  $j = 1, \dots, n$ ,  $\forall t \geq 0$ , and the closed-loop trivial solution  $\bar{q}(t) \equiv 0_n$  is:

1. globally finite-time stable if  $r_2 < r_1$ ;
2. globally asymptotically stable with (local) exponential stability if  $r_2 = r_1$ .

**Proof.** Since the proposed control scheme is applied—with all its previously stated specifications—Proposition 1 holds and consequently  $|\tau_j(t)| = |u_j(t)| < T_j$ ,  $j = 1, \dots, n$ ,  $\forall t \geq 0$ . Then, all that remains to be proven is that the additional considerations give rise to the specific stability properties claimed in items 1

and 2 of the statement. In this direction, let  $\hat{r}_i = (r_{i1}, \dots, r_{in})^T$ ,  $i = 1, 2, 3$ ,  $r = (\hat{r}_1^T, \hat{r}_2^T, \hat{r}_3^T)^T$ ,  $D \triangleq \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : K_i x_i \in D_{i1} \times \dots \times D_{in}, i = 1, 2, 3\} = \{(x_1, x_2, x_3) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : |x_{ij}| < L_{ij}/k_{ij}, i = 1, 2, 3, j = 1, \dots, n\}$ , and consider the previously defined state (vector) variables and the consequent closed-loop state-space representation  $\dot{x} = f(x) + \hat{f}(x)$ , with  $f$  and  $\hat{f}$  as defined through Eq. (18). Since  $D$  defines an open neighborhood of the origin, there exists  $\rho > 0$  such that  $B_\rho \triangleq \{x \in \mathbb{R}^{3n} : \|x\| < \rho\} \subset D$ . Moreover, for every  $x \in B_\rho$  and all  $\varepsilon \in (0, 1]$ , we have that  $\delta_\varepsilon^r(x) \in B_\rho$  (since  $\|\delta_\varepsilon^r(x)\| < \|x\|$ ,  $\forall \varepsilon \in (0, 1)$ ), and, for every  $j \in \{1, \dots, n\}$ ,

$$f_j(\delta_\varepsilon^r(x)) = \varepsilon^{r_{2j}} x_{2j} = \varepsilon^{r_2} x_{2j} = \varepsilon^{(r_2 - r_1) + r_1} x_{2j} = \varepsilon^{(r_2 - r_1) + r_{1j}} f_{j1}(x)$$

$$\begin{aligned} f_{n+j}(\delta_\varepsilon^r(x)) &= -H_j^{-1}(q_d) [s_1(K_1 \delta_\varepsilon^{r_1}(x_1)) + s_2(K_2 \delta_\varepsilon^{r_3}(x_3))] \\ &= -H_j^{-1}(q_d) [s_1(\varepsilon^{r_1} K_1 x_1) + s_2(\varepsilon^{r_3} K_2 x_3)] \\ &= -H_j^{-1}(q_d) [\varepsilon^{\alpha_1} s_1(K_1 x_1) + \varepsilon^{\alpha_2} s_2(K_2 x_3)] \\ &= -H_j^{-1}(q_d) \varepsilon^{2r_2 - r_1} [s_1(K_1 x_1) + s_2(K_2 x_3)] \\ &= -\varepsilon^{(r_2 - r_1) + r_2} H_j^{-1}(q_d) [s_1(K_1 x_1) + s_2(K_2 x_3)] \\ &= \varepsilon^{(r_2 - r_1) + r_{2j}} f_{n+j}(x) \end{aligned} \quad (21)$$

$$\begin{aligned} f_{2n+j}(\delta_\varepsilon^r(x)) &= -As_3(\delta_\varepsilon^{r_3}(x_3)) + B\delta_\varepsilon^{r_2}(x_2) \\ &= -As_3(\varepsilon^{r_3} x_3) + \varepsilon^{r_2} Bx_2 \\ &= -A\varepsilon^{\alpha_3} s_3(x_3) + \varepsilon^{r_2} Bx_2 \\ &= \varepsilon^{r_2} [-As_3(x_3) + Bx_2] \\ &= \varepsilon^{(r_2 - r_3) + r_3} [-As_3(x_3) + Bx_2] \\ &= \varepsilon^{(r_2 - r_1) + r_{3j}} f_{2n+j}(x) \end{aligned}$$

whence one concludes that  $f$  is a locally  $r$ -homogeneous vector field of degree  $\alpha = r_2 - r_1$ , with domain of homogeneity  $B_\rho$ . Hence, by Theorems 1 and 3, Lemma 1 and Remark 4, the origin of the state equation  $\dot{x} = f(x)$  is concluded to be a globally finite-time stable equilibrium if  $r_2 < r_1$ , and a globally asymptotically stable equilibrium with (local) exponential stability if  $r_2 = r_1$ . Thus, by Theorem 3, Lemma 2, and Remarks 2 and 5, the origin of the closed-loop system  $\dot{x} = f(x) + \hat{f}(x)$  is concluded to be a globally finite-time stable equilibrium provided that  $r_2 < r_1$ , and a globally asymptotically stable equilibrium with (local) exponential stability provided that  $r_2 = r_1$ , if

$$\begin{aligned} \mathcal{L}_0 &\triangleq \lim_{\varepsilon \rightarrow 0^+} \left\| \varepsilon^{-\alpha} \text{diag}[\varepsilon^{-r_{11}}, \dots, \varepsilon^{-r_{1n}}, \varepsilon^{-r_{21}}, \dots, \varepsilon^{-r_{2n}}, \varepsilon^{-r_{31}}, \dots, \varepsilon^{-r_{3n}}] \hat{f}(\delta_\varepsilon^r(x)) \right\| \\ &= \lim_{\varepsilon \rightarrow 0^+} \left\| \varepsilon^{-\alpha} \text{diag}[\varepsilon^{-r_{21}}, \dots, \varepsilon^{-r_{2n}}] [\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))] \right\|^T \\ &= \lim_{\varepsilon \rightarrow 0^+} \left\| \varepsilon^{-\alpha - r_2} [\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))] \right\|^T \\ &= \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{r_1 - 2r_2} \left\| [\hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x))] \right\|^T \\ &= 0 \end{aligned} \quad (22)$$

for all  $x \in S_c^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\| = c\}$ , resp.  $x \in S_{r,c}^{3n-1} = \{x \in \mathbb{R}^{3n} : \|x\|_r = c\}$ , for some  $c > 0$  such that  $S_c^{3n-1} \subset D$ , resp.  $S_{r,c}^{3n-1} \subset D$ . Hence, from (18b), under the consideration of Property 2.2 and Remark 1, we have, for all such  $x \in S_c^{3n-1}$ , resp.  $x \in S_{r,c}^{3n-1}$ :

$$\begin{aligned}
 & \left\| \left[ \hat{f}_{n+1}(\delta_\varepsilon^r(x)), \dots, \hat{f}_{2n}(\delta_\varepsilon^r(x)) \right]^T \right\| \\
 &= \left\| -H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, \varepsilon^{r_2}x_2)\varepsilon^{r_2}x_2 \right. \\
 &\quad \left. - \mathcal{H}(\varepsilon^{r_1}x_1) \left[ s_1(\varepsilon^{r_1}K_1x_1) + s_2(\varepsilon^{r_3}K_2x_3) \right] \right\| \\
 &\leq \left\| -H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)\varepsilon^{2r_2}x_2 \right\| \\
 &\quad + \left\| \mathcal{H}(\varepsilon^{r_1}x_1) \left[ \varepsilon^{\alpha_1}s_1(K_1x_1) + \varepsilon^{\alpha_2}s_2(K_2x_3) \right] \right\| \\
 &\leq \left\| -\varepsilon^{2r_2}H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \right\| \\
 &\quad + \left\| \mathcal{H}(\varepsilon^{r_1}x_1)\varepsilon^{2r_2-r_1} \left[ s_1(K_1x_1) + s_2(K_2x_3) \right] \right\| \\
 &\leq \varepsilon^{2r_2} \left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \right\| \\
 &\quad + \varepsilon^{2r_2-r_1} \left\| \mathcal{H}(\varepsilon^{r_1}x_1) \left[ s_1(K_1x_1) + s_2(K_2x_3) \right] \right\|
 \end{aligned}$$

and consequently, from (22), we get

$$\begin{aligned}
 \mathcal{L}_0 &\leq \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{r_1} \left\| H^{-1}(\varepsilon^{r_1}x_1 + q_d)C(\varepsilon^{r_1}x_1 + q_d, x_2)x_2 \right\| \\
 &\quad + \lim_{\varepsilon \rightarrow 0^+} \left\| \mathcal{H}(\varepsilon^{r_1}x_1) \left[ s_1(K_1x_1) + s_2(K_2x_3) \right] \right\| \\
 &\leq \left\| H^{-1}(q_d)C(q_d, x_2)x_2 \right\| \lim_{\varepsilon \rightarrow 0^+} \varepsilon^{r_1} \\
 &\quad + \left\| s_1(K_1x_1) + s_2(K_2x_3) \right\| \lim_{\varepsilon \rightarrow 0^+} \left\| \mathcal{H}(\varepsilon^{r_1}x_1) \right\| \\
 &\leq \left\| s_1(K_1x_1) + s_2(K_2x_3) \right\| \cdot \left\| \mathcal{H}(0_n) \right\| = 0
 \end{aligned}$$

(note, from (19), that  $\|\mathcal{H}(0_n)\| = \|H^{-1}(q_d) - H^{-1}(q_d)\| = 0$ ), which completes the proof.  $\square$

**Corollary 1.** Consider the proposed control scheme taking  $\sigma_{ij}$ ,  $i = 1, 2, 3$ ,  $j = 1, \dots, n$ , such that

$$\sigma_{ij}(\varsigma) = \text{sign}(\varsigma) |\varsigma|^{\beta_{ij}} \quad \forall |\varsigma| \leq L_{ij} \in (0, \infty] \quad (23)$$

with –for every  $j = 1, \dots, n$ – constants  $\beta_{ij} = \beta_i$ ,  $i = 1, 2, 3$ , such that

$$0 < \beta_1 \leq 1, \quad \beta_2 = \beta_1, \quad \beta_3 = \frac{1 + \beta_1}{2} \quad (24)$$

Thus, for any positive definite diagonal matrices  $K_1$ ,  $K_2$ ,  $A$  and  $B$ ,  $|\tau_j(t)| = |u_j(t)| < T_j$ ,  $j = 1, \dots, n$ ,  $\forall t \geq 0$ , and the closed-loop trivial solution  $\bar{q}(t) \equiv 0_n$  is:

1. globally finite-time stable if  $0 < \beta_1 < 1$ ;
2. globally asymptotically stable with (local) exponential stability if  $\beta_1 = 1$ .

**Proof.** Note that, given any  $r_{ij} > 0$ , for every  $\varsigma \in (-L_{ij}, L_{ij})$ :  $\varepsilon^{r_{ij}}\varsigma \in (-L_{ij}, L_{ij})$  and  $\sigma_{ij}(\varepsilon^{r_{ij}}\varsigma) = \text{sign}(\varepsilon^{r_{ij}}\varsigma) |\varepsilon^{r_{ij}}\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}} \text{sign}(\varsigma) |\varsigma|^{\beta_{ij}} = \varepsilon^{r_{ij}\beta_{ij}} \sigma_{ij}(\varsigma)$ ,  $\forall \varepsilon \in (0, 1]$ . Hence, under the consideration of expressions (24), for every  $j \in \{1, \dots, n\}$ , we have, for any  $r_{1j} = r_1 > 0$ , that taking  $r_{2j} = r_2 = (1 + \beta_1)r_1/2$  and  $r_{3j} = r_3 = r_1$ ,  $\sigma_{ij}$ ,  $i = 1, 2$ , are locally  $r_i$ -homogeneous of degree  $\alpha_{1j} = \alpha_1 = r_1\beta_1 = r_3\beta_2 = \alpha_2 = \alpha_{2j}$  with domain of homogeneity  $D_{ij} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{ij}\}$ , and  $\sigma_{3j}$  is locally  $r_1$ -homogeneous of degree  $\alpha_{3j} = \alpha_3 = (1 + \beta_1)r_3/2 = (1 + \beta_1)r_1/2 = r_2$  with domain of homogeneity  $D_{3j} = \{\varsigma \in \mathbb{R} : |\varsigma| < L_{3j}\}$ , while  $0 < \beta_1 \leq 1 \Rightarrow \beta_1 > 0 \geq \beta_1 - 1 \geq (\beta_1 - 1)/2 \Rightarrow (\beta_1 + 1)/2 \leq 1 < 1 + \beta_1 \Leftrightarrow (\beta_1 + 1)r_1/2 \leq r_1 < (1 + \beta_1)r_1 \Leftrightarrow r_2 \leq r_1 < 2r_2 \Leftrightarrow r_2 - r_1 \leq 0 < 2r_2 - r_1$ . The requirements of Proposition 2 are thus concluded to be satisfied with  $0 < \beta_1 < 1 \Rightarrow r_2 < r_1$  and  $\beta_1 = 1 \Rightarrow r_2 = r_1$ .  $\square$

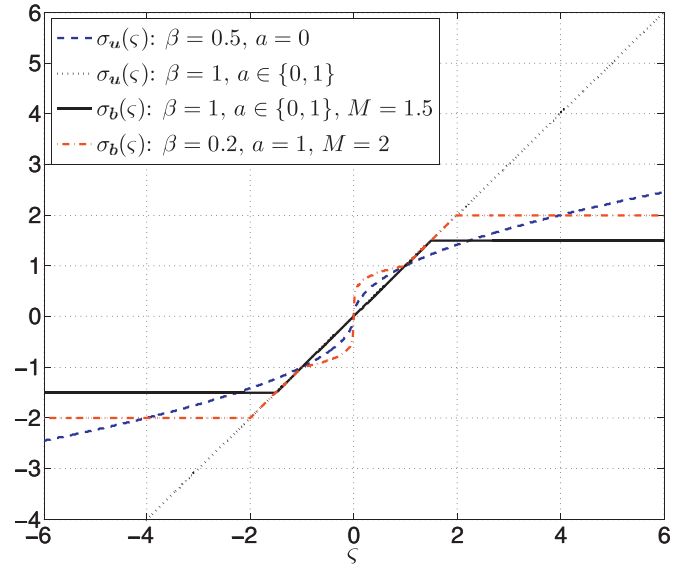


Fig. 2. Examples of  $\sigma_u(\varsigma; \beta, a)$  and  $\sigma_b(\varsigma; \beta, a, M)$ .

**Remark 11.** Since the results of this section depart from the application of the proposed control scheme, the cases of Proposition 2 with  $r_2 > r_1$  and Corollary 1 with  $\beta_1 > 1$  are particular cases of Proposition 1 where the closed-loop trivial solution  $\bar{q}(t) \equiv 0_n$  is globally asymptotically stable but not (locally) exponentially stable (in accordance to Footnote 4).

### 5. Simulation results

The proposed scheme was implemented through computer simulations considering the model of a 2-DOF mechanical manipulator corresponding to the experimental robotic arm used in [41]. For such a robot, the various terms characterizing the system dynamics in Eq. (4) are given by

$$H(q) = \begin{pmatrix} 2.351 + 0.168 \cos q_2 & 0.102 + 0.084 \cos q_2 \\ 0.102 + 0.084 \cos q_2 & 0.102 \end{pmatrix}$$

$$C(q, \dot{q}) = \begin{pmatrix} -0.084\dot{q}_2 \sin q_2 & -0.084(\dot{q}_1 + \dot{q}_2) \sin q_2 \\ 0.084\dot{q}_1 \sin q_2 & 0 \end{pmatrix}$$

$$g(q) = \begin{pmatrix} 38.465 \sin q_1 + 1.825 \sin(q_1 + q_2) \\ 1.825 \sin(q_1 + q_2) \end{pmatrix}$$

Assumption 1 is thus satisfied with  $B_{g1} = 40.29$  Nm and  $B_{g2} = 1.825$  Nm. Furthermore, the input saturation bounds are  $T_1 = 150$  Nm and  $T_2 = 15$  Nm for the first and second links, respectively, whence one can corroborate that Assumption 2 is fulfilled too. For the sake of simplicity, units will be subsequently omitted.

For the application of the proposed design methodology, let us define the functions

$$\sigma_u(\varsigma; \beta, a) = \text{sign}(\varsigma) \max\{|\varsigma|^\beta, a|\varsigma|\} \quad (25a)$$

$$\sigma_b(\varsigma; \beta, a, M) = \text{sign}(\varsigma) \min\{|\sigma_u(\varsigma; \beta, a)|, M\} \quad (25b)$$

for constants  $\beta > 0$ ,  $a \in \{0, 1\}$  and  $M > 0$  (other function definitions, that may be used in this context, are presented in [42]). Fig. 2 shows examples.

Based on the functions in Eq. (25), we define –for every  $j = 1, 2$ – those involved in the implementations performed in this section as

$$\sigma_{ij}(\varsigma) = \sigma_b(\varsigma; \beta_i, a_{ij}, M_{ij}) \quad i = 1, 2 \quad (26a)$$



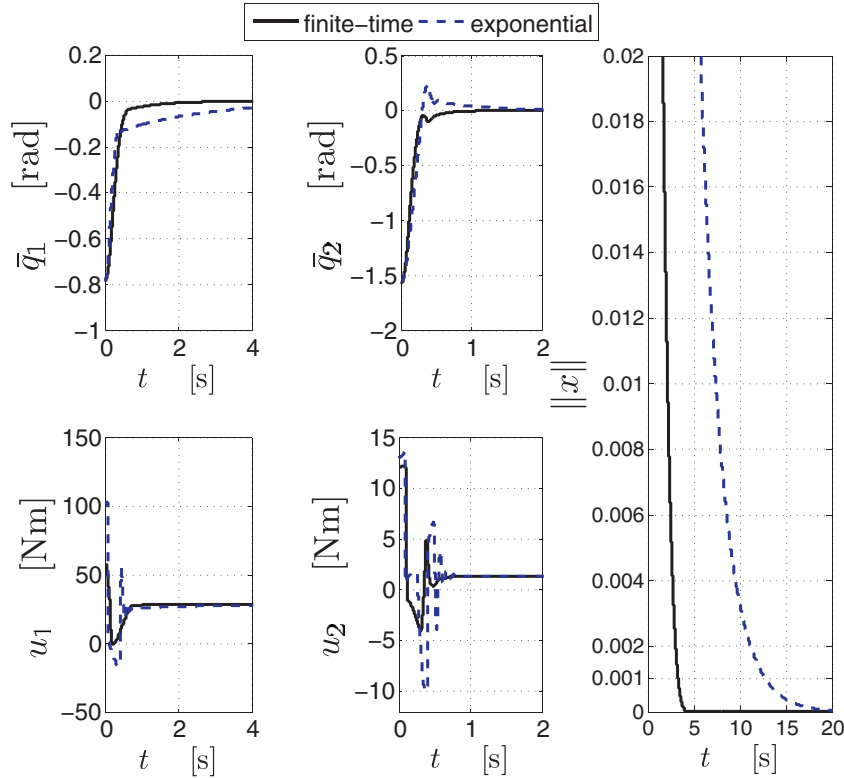


Fig. 3. Finite-time vs exponential stabilization: position errors ( $\uparrow$ ), control signals ( $\downarrow$ ) and  $\|x(t)\|$  ( $\rightarrow$ ).

$$\sigma_{3j}(\zeta) = \sigma_u(\zeta; \beta_3, a_{3j}) \quad (26b)$$

Let us note that through these definitions we have  $B_j = M_{1j} + M_{2j}$ ,  $j = 1, 2$  (see (16) and recall footnote<sup>8</sup>). Thus, by fixing  $M_{11} = M_{21} = 50$  and  $M_{12} = M_{22} = 6.4$ , the inequalities from expression (16) are satisfied. The implementations were run taking the desired configuration at  $q_d = (\pi/4 \ \pi/2)^T$  [rad] and initial conditions as  $q(0) = \dot{q}(0) = \vartheta_c(0) = 0_2$ .

Following the design procedure in accordance to Corollary 1, we began by a test where the aim is to corroborate the convergence difference among the closed-loop trajectories obtained with the proposed finite-time controller, taking  $\beta_1 = \beta_2 = 1/2$  and  $\beta_3 = 3/4$ , and the analog exponential stabilizer, i.e. with  $\beta_1 = \beta_2 = \beta_3 = 1$ . All the rest, including control and auxiliary subsystem gains, remained unchanged. For this test we took  $a_{ij} = 0$ ,  $i = 1, 2, 3$ ,  $j = 1, 2$ . As a performance comparison indicator, we obtained the  $q$ -stabilization time  $t_\rho^s$ , defined as  $t_\rho^s \triangleq \inf\{t_s \geq 0 : \|x(t)\| \leq \rho \ \forall t \geq t_s\}$ , where  $x \triangleq (\bar{q}^T \ \dot{q}^T \ \vartheta^T)^T$ .

Fig. 3 shows results obtained taking  $K_1 = K_2 = \text{diag}[70, 20]$ ,  $A = \text{diag}[30, 30]$  and  $B = \text{diag}[70, 20]$ . One sees that the stabilization objective was achieved by both controllers avoiding input saturation. Moreover, the contrast among the different types of trajectory convergence, in accordance to the corresponding controller nature, is clear from the graphs. In particular, one sees that, with the finite-time controller, the position errors, and actually the (norm of the) whole state vector in the extended state space ( $x = (\bar{q}^T \ \dot{q}^T \ \vartheta^T)^T$ ), converge to zero in less than 5 s, remaining invariant thereafter. The exponential controller, instead, generated asymptotically convergent closed-loop trajectories with longer stabilization time. In terms of the  $\rho$ -stabilization time for  $\rho = 0.01$ , we obtained  $t_{0.01}^s = 7.38$  s for the exponential controller vs  $t_{0.01}^s = 2.16$  s for the finite-time stabilizer. Let us note that, in view of the different types of trajectory convergence, whatever the control parameter tuning be, there will always be a sufficiently small value

$\rho^*$  such that  $t_\rho^s$  is smaller in the finite-time controller case for all  $\rho < \rho^*$ . The control gain tuning was fixed so as to render such a convergence difference visibly clear from the graphs.

Another test was run in order to compare the finite-time controller proposed here with an observer-based algorithm from [16]. More precisely, with the control replacement posed in the last paragraph of [16, Section 3], having the form

$$u = g(q) - K_1 \text{Sat}(\text{Sig}(\zeta_1; \beta_1)) - K_2 \text{Sat}(\text{Sig}(\zeta_2; \beta_2)) \quad (27)$$

where, for any  $x \in \mathbb{R}^n$  and  $\beta \in \mathbb{R}_{>0}$ ,  $\text{Sat}(x) = [\text{sat}(x_1), \dots, \text{sat}(x_n)]^T$ ,  $\text{Sig}(x; \beta) = [\text{sign}(x_1)|x_1|^\beta, \dots, \text{sign}(x_n)|x_n|^\beta]^T$ ,  $\beta_1 \in (0, 1)$ ,  $\beta_2 = 2\beta_1/(1 + \beta_1)$ ,  $K_i = \text{diag}[k_{i1}, \dots, k_{in}]$ ,  $i = 1, 2$ , are positive definite –control gain– matrices such that

$$k_{1j} + k_{2j} < T_j - B_{gj} \quad (28)$$

$j = 1, \dots, n$  (so as to guarantee input saturation avoidance), and  $\zeta_i \in \mathbb{R}^n$ ,  $i = 1, 2$ , are the state vector variables of a finite-time observer with the following state-space representation

$$\dot{\zeta}_1 = \zeta_2 - L_1 \text{Sig}(\zeta_1 - \bar{q}; \alpha_1) \quad (29a)$$

$$\dot{\zeta}_2 = v - L_2 \text{Sig}(\zeta_1 - \bar{q}; \alpha_2) \quad (29b)$$

where

$$v = M^{-1}(q) [-g(q) - C(q, \zeta_2)\zeta_2 + u] \quad (30)$$

$\alpha_1 = (1 + \beta_1)/2$ ,  $\alpha_2 = \beta_1$ ,  $L_i$ ,  $i = 1, 2$ , are positive definite diagonal –observer gain– matrices, and the position error vector variable  $\bar{q}$  is as previously defined. In view of the different degree of dependence on the system model among the control scheme in expressions (27)–(30) –subsequently designated as H02– and the controller proposed here, the motivation of this new test is to carry out the comparison under system parameter uncertainties. To account for such parameter imprecisions, the test was run replacing in the corresponding control algorithms  $g(\cdot)$ ,  $C(\cdot)$  and/or

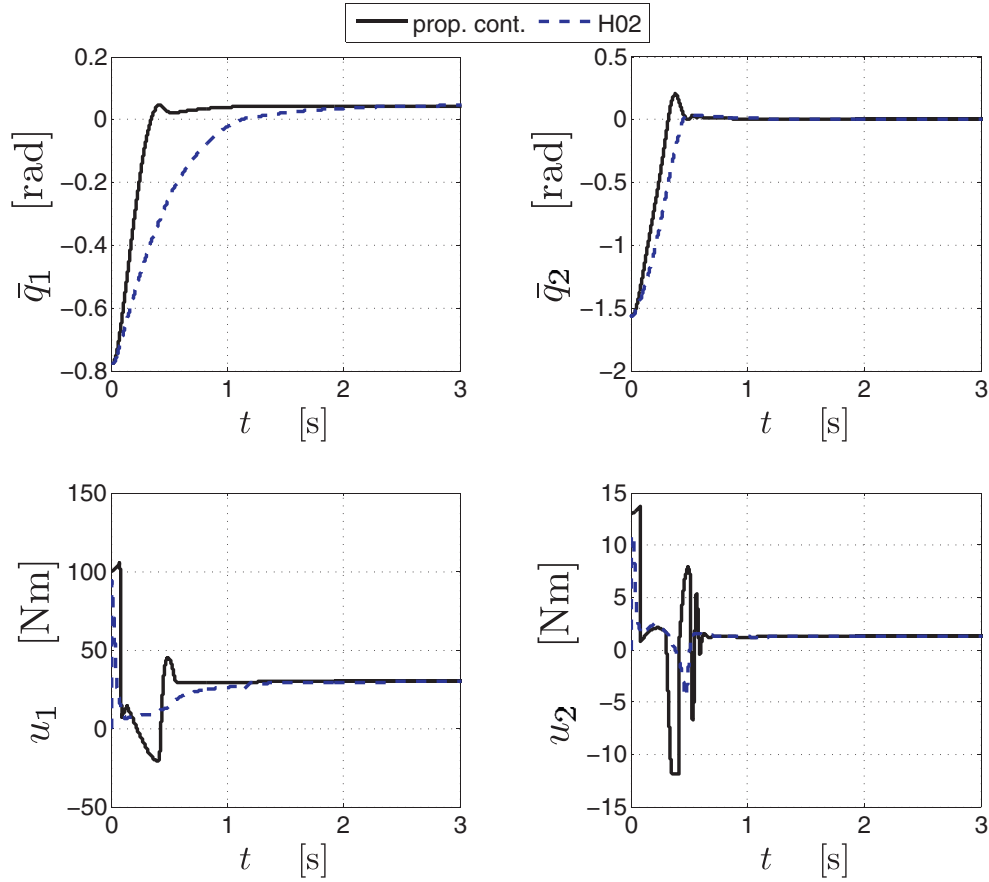


Fig. 4. Proposed finite-time controller vs H02 with biased parameter estimations: position errors ( $\uparrow$ ) and control signals ( $\downarrow$ ).

$M(\cdot)$  by  $\hat{g}(\cdot) = k_g g(\cdot)$ ,  $\hat{C}(\cdot, \cdot) = k_C C(\cdot, \cdot)$  and  $\hat{M}(\cdot) = k_M M(\cdot)$ , respectively, taking  $k_g = k_C = k_M = 1.15$ . Furthermore, the test was implemented keeping the same auxiliary functions for the proposed finite-time controller, i.e. those in Eq. (26) with  $M_{11} = M_{21} = 50$  and  $M_{12} = M_{22} = 6.4$ , and  $\beta_1 = 1/2$  for both controllers (i.e.  $\beta_1 = \beta_2 = 1/2$  and  $\beta_3 = 3/4$  for the proposed finite-time scheme, and  $\beta_1 = \alpha_2 = 1/2$ ,  $\beta_2 = 2/3$  and  $\alpha_1 = 3/4$  for the H02 controller). The same desired configuration as in the previous test and analog initial conditions were also taken, i.e.  $q_d = (\pi/4 \ \pi/2)^T$  and  $q(0) = \dot{q}(0) = 0_2$  for both controllers,  $\vartheta_c(0) = 0_2$  for the proposed finite-time scheme and  $\zeta_1(0) = \zeta_2(0) = 0_2$  for the H02 controller. As a variation with respect to the previous implementations, for this test the proposed finite-time algorithm was run taking  $a_{ij} = 1$ ,  $i = 1, 2, 3$ ,  $j = 1, 2$ . The control parameters were tuned, after numerous simulations, so as to get the best possible performance for every controller, taking into account the advantages and design features of each one, in particular, with high enough observer gains in the H02 case to considerably reduce the uncertainty effects in the observer estimations, and taking into account its (saturation-avoidance) control gain constraints (28), and on the other hand the liberty to fix any desired (positive) control gain value in the case of the control scheme proposed here. The resulting values were  $K_1 = \text{diag}[100, 50]$ ,  $K_2 = \text{diag}[80, 45]$ ,  $A = \text{diag}[31, 31]$ ,  $B = \text{diag}[35, 20]$  for the proposed scheme, and  $K_1 = \text{diag}[55, 6]$ ,  $K_2 = \text{diag}[45, 5]$ ,  $L_1 = L_2 = \text{diag}[1000, 250]$  for the H02 controller. Furthermore, in view of the considered system uncertainties, as performance comparison indicators, we calculated a modified version of the  $\varrho$ -stabilization time defined (for this test) as  $t_\varrho^s \triangleq \inf\{t_s \geq 0 : \|\dot{q}(t)\| \leq \varrho \ \forall t \geq t_s\}$ , as well as the Integral of the Square of the Error (ISE) index, defined as  $\int_{t_0}^{t_0+\Delta} \|x(t)\|^2 dt$ , applied during the steady-state phase, more precisely with  $t_0 = t_\rho^s$  and common  $\Delta$  for every con-

Table 1  
Performance indices: proposed controller vs H02.

	Prop. cont.	H02
$t_{0.01}^s$	1.21	2.37
ISE	0.015	0.069

troller, both in their respective extended state space, i.e. with  $x = (\bar{q}^T \ \dot{q}^T \ \vartheta^T)^T$  for the proposed scheme, and  $x = (\bar{q}^T \ \dot{q}^T \ \zeta_1^T \ \zeta_2^T)^T$  in the case of the H02 controller.

Fig. 4 shows the results obtained for this test. One corroborates that the system trajectories reached an equilibrium avoiding input saturation, both controllers with a reduced steady-state error. Table 1 shows the resulting values for each one of the considered performance indices –with the ISE calculated taking  $t_0 = t_{0.01}^s$  and  $\Delta = 7.6$  s– whence one sees that the proposed controller achieved a faster response and a lower ISE index value.

As a suggestion from an anonymous reviewer, under the same parameter uncertainty considered in the previous test, we further performed an additional test where the (discontinuous) algorithms from [24] and [11] were involved. More precisely, leaving the controller output from the scheme proposed in this work, and replacing the auxiliary (dynamic dissipation) subsystem in Eq. (15) by the respectively referred algorithms, i.e. by Eq. (29) with  $\alpha_1 = 1/2$  and  $\alpha_2 = 0$  in both cases, taking  $\nu = 0_n$  and  $\vartheta = \zeta_2 - L_1 \text{Sig}(\zeta_1 - \bar{q}; 1/2)$  in the case of the differentiator from [24], subsequently designated as L98, and taking  $\nu$  as in (30) and  $\vartheta = \zeta_2$  in the case of the observer from [11], subsequently designated as DFL05. This time, the control gain parameter values were kept for every one of the tested controller, and the auxiliary subsystem parameters

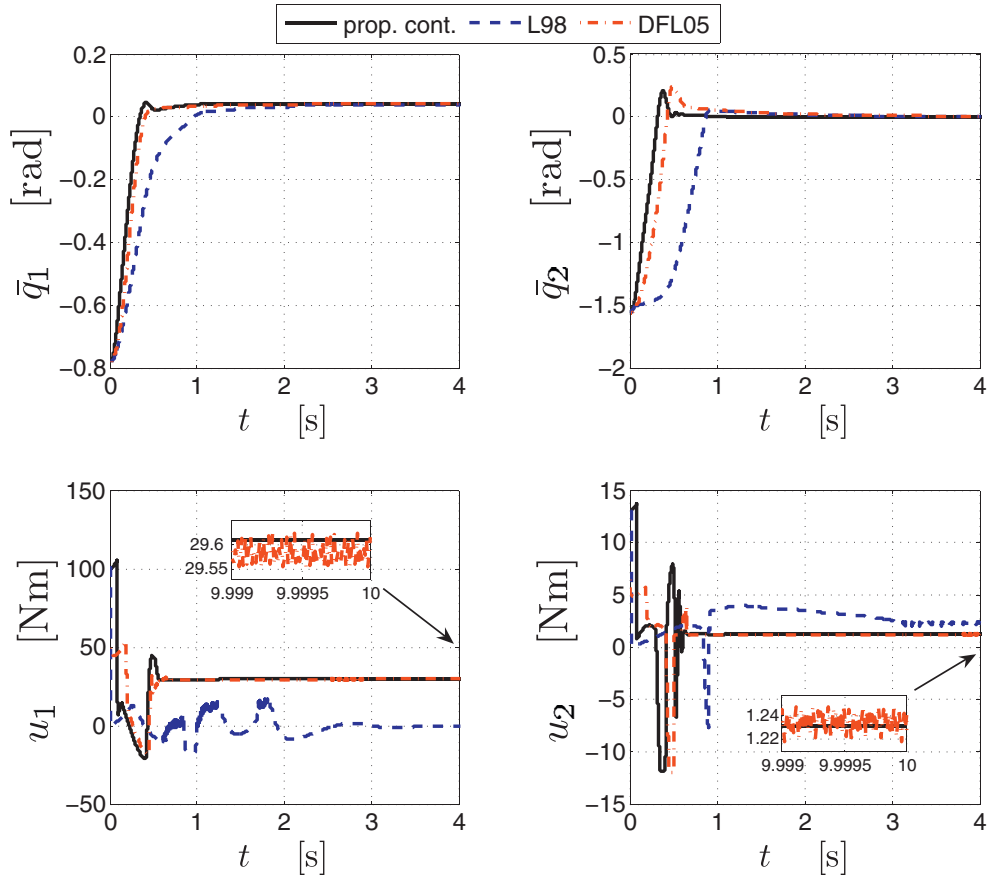


Fig. 5. Proposed finite-time controller vs L98 and DFL05 with biased parameter estimations: position errors ( $\uparrow$ ) and control signals ( $\downarrow$ ).

**Table 2**  
Performance indices: proposed controller vs L98 and DFL05.

	Prop. cont.	L98	DFL05
$t_{0,01}^s$	1.21	2.89	2.51
ISE	0.014	0.473	0.013

were fixed (after numerous simulations) so as to obtain the best possible closed-loop performance at every one of the implemented cases. The common control gains were  $K_1 = \text{diag}[100, 50]$  and  $K_2 = \text{diag}[80, 45]$ , and the resulting auxiliary subsystem parameter values were  $L_1 = \text{diag}[502.02, 152.02]$  and  $L_2 = \text{diag}[100, 100]$  in the case of L98,  $L_1 = L_2 = \text{diag}[13.2, 5.2]$  in the case of DFL05, and those from the previous test for the proposed controller, *i.e.*  $A = \text{diag}[31, 31]$  and  $B = \text{diag}[35, 20]$ . The same performance indices defined for the previous test were considered for this test.

Fig. 5 shows the results obtained for this test. The graphs show that at all the implemented cases an equilibrium is attained entailing a reduced steady-state error. Table 2 shows the resulting values for each one of the considered performance indices —with the ISE calculated taking  $t_0 = t_{0,01}^s$  and  $\Delta = 7.1$  s— whence one sees that the proposed controller achieved the fastest response, and L98 produced the highest ISE index value, with the other two controllers being very close at this latter aspect. Let us further add that each one of the implementations were reproduced using different integration steps. The responses obtained with the proposed controller and with DFL05 did not produce perceptible changes through such a technical modification in the numerical simulations. Such was not the case for L98. The results shown in Fig. 5 were obtained using an integration step of  $10^{-5}$  s. When they were reproduced

with an integration step of  $10^{-4}$  s, the system response obtained with L98 was notoriously different with respect to the corresponding one reported in Fig. 5 (it actually kept oscillating). This leads us to conclude that L98 is sensible to technical implementation aspects such as the integration step or the involved numerical-integration algorithm (recall the dynamic nature of the auxiliary subsystem) or the sampling period (under digital or computer-based implementations, as commonly done nowadays), particularly if the former is attached to the latter. Furthermore, the implementation involving the DFL05 algorithm produced a(n although reduced but existent) chattering type effect on the control signals, as actually shown in Fig. 5. Such phenomena noticed in the L98 and DFL05 cases are related to their discontinuous nature and are avoided in the case of the proposed controller.

## 6. Conclusions

Global output-feedback stabilization of mechanical systems with input constraints guaranteeing finite-time or exponential stabilization has been made possible through local homogeneity. A continuous control scheme based on such a concept has been thoroughly developed and formally proposed, leaving the designer the election on the mentioned types of convergence through a simple parameter. The proposed scheme achieves the control objective through a continuous dynamic dissipator with a simple generalized structure. The work has been complemented through a simulation implementation section where it has not only been possible to illustrate the application of the proposed method and confirm the analytical results but also to corroborate the performance difference with respect to previous observer-based and

differentiation algorithms. Future work will focus on a more thorough robustness study under uncertainties.

### Acknowledgments

The authors greatly appreciate the financial support by CONACYT, Mexico; grant number: CB-2014-01-239833.

### Appendix A. Passive systems

We recall here the definition of a passive dynamical system represented by state model [21, §6.2]

$$\dot{x} = f(x, u) \quad (\text{A.1a})$$

$$y = h(x, u) \quad (\text{A.1b})$$

with  $f: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $h: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  being continuous,  $f(x, u)$  locally Lipschitz on  $\mathbb{R}^n \times \mathbb{R}^m \setminus (0_n, 0_m)$ ,  $f(0_n, 0_m) = 0_n$  and  $h(0_n, 0_m) = 0_m$ .

**Definition 8.** The system represented by the state model in Eqs. (A.1) is said to be *passive* if there exists a continuously differentiable positive semidefinite function  $V(x)$  (called the *storage function*) such that

$$\dot{V}(x, u) = \frac{\partial V}{\partial x} f(x, u) \leq u^T y$$

$\forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$ . Moreover, it is said to be

- *lossless* if  $\dot{V}(x, u) = u^T y$ ;
- *input strictly passive* if  $\dot{V}(x, u) \leq u^T y - u^T \varphi(u)$  for some function  $\varphi: \mathbb{R}^m \rightarrow \mathbb{R}^m$  such that  $u^T \varphi(u) > 0, \forall u \neq 0_m$ ;
- *output strictly passive* if  $\dot{V}(x, u) \leq u^T y - y^T \rho(y)$  for some function  $\rho: \mathbb{R}^m \rightarrow \mathbb{R}^m$  such that  $y^T \rho(y) > 0, \forall y \neq 0_m$ ;
- *strictly passive* if  $\dot{V}(x, u) \leq u^T y - \psi(x)$  for some positive definite function  $\psi: \mathbb{R}^n \rightarrow \mathbb{R}$ .

**Definition 9.** The system represented by the state model in Eq. (A.1) is said to be *zero-state observable*, if no solution of  $\dot{x} = f(x, 0_m)$  can stay identically in  $S = \{x \in \mathbb{R}^n : h(x, 0_m) = 0_m\}$ , other than the trivial solution  $x(t) \equiv 0_n$  (or equivalently  $u(t) \equiv y(t) \equiv 0_m \Rightarrow x(t) \equiv 0_n$ ).

### Appendix B. Proof of Theorem 2

Let  $V_i(x_i)$  and  $V_j(x_j)$  be the storage functions for  $\Sigma_i$  and  $\Sigma_j$ , respectively. As proven in [21, Lemma 6.7],  $V_j(x_j)$  is positive definite. Take  $V(x_1, x_2) = V_i(x_i) + V_j(x_j)$  as Lyapunov function candidate for the closed loop. Its derivative along the system trajectories  $\dot{V}$  satisfies

$$\dot{V} = \frac{\partial V_i}{\partial x_i} f_i(x_i, e_i) + \frac{\partial V_j}{\partial x_j} f_j(x_j, e_j) \leq e_i^T y_i + e_j^T y_j - \psi(x_j)$$

with  $\psi$  being positive definite in its argument, i.e.  $\psi(x_j) > 0, \forall x_j \neq 0_{n_j}$ , and  $\psi(0_{n_j}) = 0$ . Since  $u_1 = u_2 = 0_m$ , it then follows that

$$\begin{aligned} \dot{V} &\leq (-1)^i y_j^T y_i + (-1)^j y_i^T y_j - \psi(x_j) \\ &\leq (-1)^i y_j^T y_i - (-1)^i y_i^T y_j - \psi(x_j) \\ &\leq -\psi(x_j) \end{aligned}$$

From the positive definite character of  $\psi$ , we have that  $S \triangleq \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \dot{V} = 0\} = \{(x_1, x_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : x_j = 0_{n_j}\}$ . It then follows that

$$\begin{aligned} (x_1, x_2)(t) \in S \quad \forall t &\Rightarrow x_j(t) \equiv 0_{n_j} \\ &\Rightarrow \dot{x}_j(t) = f_j(0_{n_j}, e_j(t)) \equiv 0_{n_j} \\ &\Rightarrow e_j(t) \equiv 0_m \end{aligned} \quad (\text{B.1a})$$

$$\Rightarrow \left\{ \begin{array}{l} y_j(t) \equiv h_j(0_{n_j}, 0_m) = 0_m \Rightarrow e_i(t) \equiv 0_m \\ y_i(t) \equiv 0_m \end{array} \right\} \Rightarrow x_i(t) \equiv 0_{n_i} \quad (\text{B.1b})$$

where the third implication, (B.1a), is a consequence of (13), and the last one, (B.1b), results from the zero-state observability of  $\Sigma_j$ . Hence,  $(x_1, x_2)(t) \equiv (0_{n_1}, 0_{n_2})$  is the only solution staying identically in  $S$ , and consequently  $\{(0_{n_1}, 0_{n_2})\}$  is the only –therefore the largest– invariant in  $S$ . Thus, from the invariance theory [29, §7.2],  $(x_1, x_2) = (0_{n_1}, 0_{n_2})$  is concluded to be asymptotically stable. Finally, radial unboundedness of  $V_1$  and  $V_2$  renders  $V$  radial unbounded, whence the concluded asymptotic stability proves to be global [29, Corollary 7.2.1] (see footnote<sup>9</sup>).

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